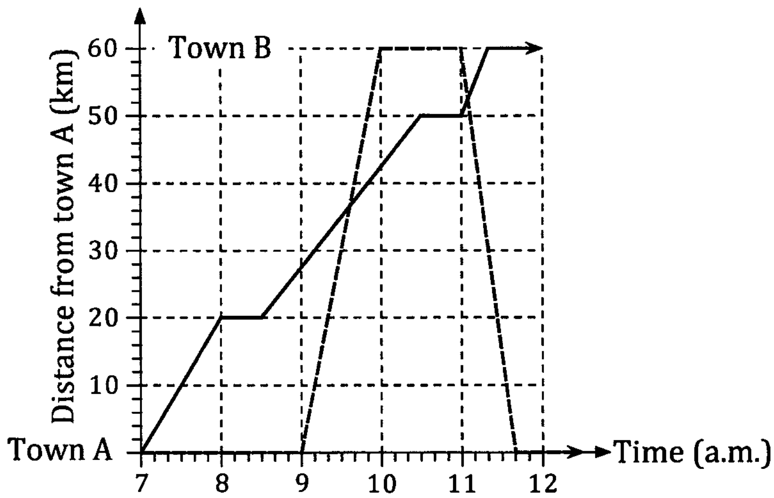
Chapter 8.

Rectilinear motion.

Situation

Two towns, A and B, are linked by straight road of length 60 km.
The distance time graph shown below is for the motion of a cyclist travelling from town A to town B, and a delivery truck making the round trip from A to B and back to A again.

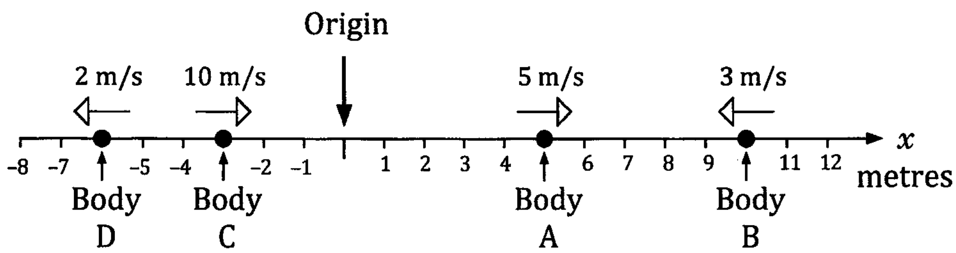


- When did the cyclist leave town A?
- When did the cyclist reach town B?
- The cyclist stopped twice for a rest. How long was each stop ?
- What speed did the cyclist maintain (a) prior to the first stop,
  (b) between the two stops,
  (c) after the second stop?
- What speed did the delivery truck maintain (a) from town A to B,
  (b) from town B back to A?
- Estimate the time and distance from A of the place where the delivery truck passed the cyclist when they were both travelling towards B.
- Estimate the time and distance from A of the place where the delivery truck passed the cyclist when the truck was returning to A.

The situation on the previous page involved a position-time graph, or travel graph. As you probably realized, the speed of the cyclist or the delivery truck could be found from the gradient of the appropriate line. Horizontal lines indicated stops, steeper lines indicated greater speed etc. The graph involved only straight lines. If curves were involved the speed could again be found from the gradient but we would then need to draw tangents or use differentiation to determine instantaneous rates of change.
Notice here that we are talking about the *speed* of something – what is the difference between speed and velocity?

Displacement, speed and velocity.
One of the commonest rates of change that concerns us is the rate at which we change our location. If we measure our location as a **displacement** from some fixed point or origin, then the rate at which we change our displacement is our **velocity**. Displacement and velocity are what are called **vector** quantities, they have magnitude (size) **and** direction. For example our displacement might be 5 kilometres north of some origin and we might be travelling with a velocity of 6 km/hour south. On the other hand distance and speed are **scalar** quantities. They have magnitude only. For example we might travel a distance of 6 km at a speed of 60 km/hour. This chapter considers only rectilinear motion – **motion in a straight line.** For motion in a straight line there are only two possible directions and these are distinguished by the use of positive and negative.

Consider the following diagram which shows the location and velocity of four objects (referred to as "bodies", but by that we do not mean cadavers!).
The origin is as shown and positive x is to the right.



The table below shows how the directions of the displacement and velocity vectors can be indicated by use of positive and negative.

Body	Distance from O	Displacement from O	Speed	Velocity
A	5 metres	5 metres	5 m/s	5 m/s
B	10 metres	10 metres	3 m/s	- 3 m/s
C	3 metres	- 3 metres	10 m/s	10 m/s
D	6 metres	- 6 metres	2 m/s	- 2 m/s

If our displacement, x metres, from some fixed origin, O, is a function of time then

$$\frac{dx}{dt}$$

gives the rate of change of this displacement with respect to time, i.e. it gives our velocity, v m/s.

For example, if $x = 5t^3 + 6t^2 + 7t + 1$

then $v = \frac{dx}{dt} = 15t^2 + 12t + 7$

Similarly, if $x = 5t^2 + 6$ then $v = 10t$

if $x = 5t^3 - 3t + 1$ then $v = 15t^2 - 3$

Example 1

A body moves in a straight line such that its displacement from an origin O, at time t seconds, is x metres where $x = t^3 + 6t + 5$.

Find the displacement and velocity when $t = 3$.

$x = t^3 + 6t + 5$

When $t = 3$, $x = (3)^3 + 6(3) + 5$

$= 50.$

$v = \frac{dx}{dt} = 3t^2 + 6.$

When $t = 3$, $v = 3(3)^2 + 6$

$= 33.$

When $t = 3$ the displacement is 50 m, and the velocity is 33 m/s

Or, using a calculator:

$t^3 + 6t + 5 \mid t = 3$	
	50
$\frac{d}{dt}(t^3 + 6t + 5) \mid t = 3$	
	33

Example 2

A body moves in a straight line such that its displacement from an origin O, at time t seconds is x metres where $x = 5t^2 + 7t + 3$.

Find (a) the displacement from O when $t = 0$,

(b) the initial (i.e. $t = 0$) velocity of the body,

(c) the value of t for which the velocity is 52 m/s,

(a) When $t = 0$ $\quad x = 5(0)^2 + 7(0) + 3$

The displacement from O when $t = 0$ is 3 m.

(b) If $x = 5t^2 + 7t + 3$ then

$$v = \frac{dx}{dt} = 10t + 7$$

Thus when $t = 0$ $\quad v = 7$

The initial velocity of the body is 7 m/s.

(c) If $v = 10t + 7$ then for $v = 52$ m/s we have

$$52 = 10t + 7$$

i.e. $$45 = 10t$$

$$t = 4{\cdot}5$$

The body has a velocity of 52 m/s when $t = 4{\cdot}5$.

Example 3

A particle is initially at an origin O. It is projected away from O and moves in a straight line such that its displacement from O, t seconds later, is x metres where $x = t(12 - t)$.

Find (a) the speed of initial projection,

(b) the distance the particle is from O when $t = 3$ and when $t = 7$,

(c) the value of t when the particle comes to rest and the distance from the origin at that time,

(d) the distance the particle travels from $t = 3$ to $t = 7$.

(a) If $x = 12t - t^2$ then

$$v = \frac{dx}{dt} = 12 - 2t$$

Thus when $t = 0$ $\quad v = 12$

The speed of projection is 12 m/s.

(b) If $t = 3$ then $x = 3(12-3)$
$= 27$

If $t = 7$ then $x = 7(12-7)$
$= 35$

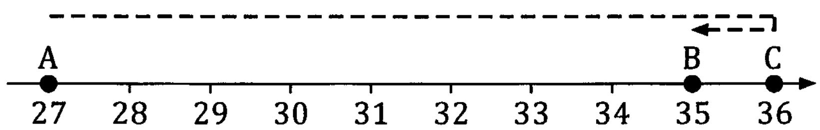
The particle is 27 m from O when $t = 3$ and 35 m from O when $t = 7$.

(c) With $v = 12 - 2t$ then the particle being at rest means $12 - 2t = 0$,
i.e. $t = 6$.

When $t = 6$, $x = 6(12-6)$
i.e. $x = 36$.

The particle is at rest when $t = 6$ and it is then 36 m from O.

(d) When $t = 3$ the particle is 27 m from O.
When $t = 7$ the particle is 35 m from O.
However the distance travelled in this time is not simply $(35 - 27)$ m. Our answer to (c) indicates that the particle stopped when $t = 6$,
i.e. at $x = 36$ m.

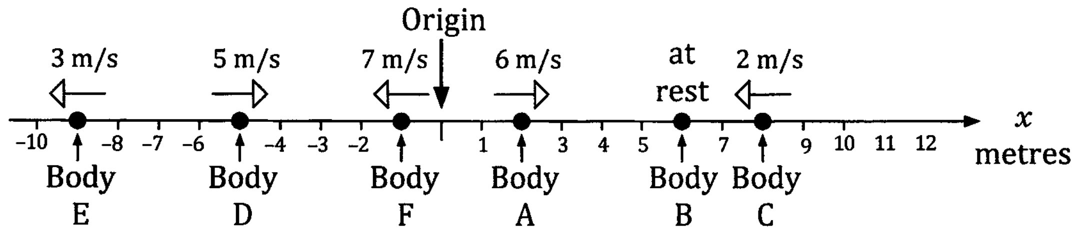


From $t = 3$ to $t = 7$ the particle travels from A to C to B, i.e. 9 m + 1 m = 10 m.
The particle travels 10 m from $t = 3$ to $t = 7$.

Exercsie 8A

1. A long straight road links three towns A, B and C with B between A and C. From town A it is 130 km to B and a further 140 km to C. A truck leaves A at 8 a.m. and travels to B. For the first half hour the truck maintains a steady speed of just 60 km/h due to speed restrictions. After this the truck is able to maintain a higher speed and arrives in town B at 9.30 a.m. Unloading and loading in town B takes one hour and then the truck travels on to C maintaining a steady 80 km/h for this part of the journey.
A car leaves A at 9 a.m. that same morning and travels directly to C. Subject to the same speed restrictions it too maintains a steady 60 km/h for the first half hour. After this first half hour the car then maintains a steady 100 km/h all the way to town C.
Draw a distance time graph for this situation and use your graph to answer the following questions:
   (a) When does each vehicle reach town C?
   (b) What steady speed did the truck maintain from 8.30 a.m. to 9.30 a.m.?
   (c) What was the average speed of the truck from A to B? (To the nearest km/h.)
   (d) When and where did the car pass the truck?

2. Copy and complete the given table for the situation shown below.



Body	Distance from O	Displacement from O	Speed	Velocity
A				
B				
C				
D				
E				
F				

3. For each of the following, x metres is the displacement of a body from an origin O, at time t seconds. Find the instantaneous velocity of the body for the given value of t.

(a) $x = t^2, \ t = 2.$ (b) $x = 2t^3 + 7t - 1, \ t = 2.$

For questions 4 → 9, x metres is the displacement of a body from an origin O at time t seconds. For each question find

(a) the initial displacement from O,
(b) the initial velocity,
(c) the speed of the body when $t = 3$.

4. $x = 3t^2 + 5t + 6$
5. $x = t(t - 3)$
6. $x = 2t^3 - 3t^2 + t$
7. $x = 6t + 3$
8. $x = 2t^2 - 20t - 3$
9. $x = 1 - 6t + 12t^2 - 8t^3$

10. The displacement of a body from an origin O, at time t seconds is x metres where

$$x = 3t^2 + 2t + 1.$$

Find the displacement and velocity and of the body when $t = 2$.

11. The displacement of a body from an origin O, at time t seconds is x metres where

$$x = 2t^3 - 3t^2 + 4t - 1.$$

Find the displacement and velocity of the body when $t = 1$.

For questions 12 → 14, x metres is the displacement of a body from an origin at time t seconds. For each question find
(a) the displacement when $t = 1$,
(b) the velocity when $t = 1$,
(c) t, (≥ 0), when the velocity is 20 m/s.

12. $x = t^2 + 6t + 1$

13. $x = t^3 - 4t^2 + 4t + 5$

14. $x = t^3 - 3t^2 + 11t + 3$

15. The displacement of a body from an origin O, at time t seconds, is x metres where

$$x = 27t + 3t^2 - \frac{t^3}{3} - 90, \quad t \geq 0.$$

Find the displacement of the body from O when the velocity is zero.

For questions 16 → 19, x metres is the displacement of a body from an origin at time t secs. For each question find
(a) the displacement when $t = 2$,
(b) the displacement when $t = 8$,
(c) t, (≥ 0), for which the body is at rest,
(d) the distance travelled from $t = 2$ to $t = 8$,
(e) the distance travelled in the 3rd second.

16. $x = 20t - t^2 + 9$

17. $x = t^2 - 8t + 20$

18. $x = 12t - t^2 + 20$

19. $x = t^3 - 6t^2 - 15t + 140$

20. A body is projected from ground level, vertically upwards, with an initial speed of 50 m/s. The body attains a height of x metres above ground level, t seconds after projection where $x = 5t(\mathrm{p} - t) + \mathrm{q}$ with p and q constant.
(a) Find p and q.
(b) Find the height and velocity of the body four seconds after projection.
(c) What will be the speed of the body when it reaches its highest point and how high will it be then?

21. A body is projected vertically upwards from ground level, with initial speed b m/s. The height it attains, t seconds after projection is x metres where x is given by

$$x = \mathrm{b}t - 5t^2.$$

Find the speed of projection if the body just reaches a height of 180 m.

22. Two particles, A and B, are travelling along the same straight line. Their displacements from an origin O, at time t seconds, are x_A and x_B where

$$x_A = 2t^2 + 3t - 6 \qquad \text{and} \qquad x_B = 30 + 60t - 3t^2.$$

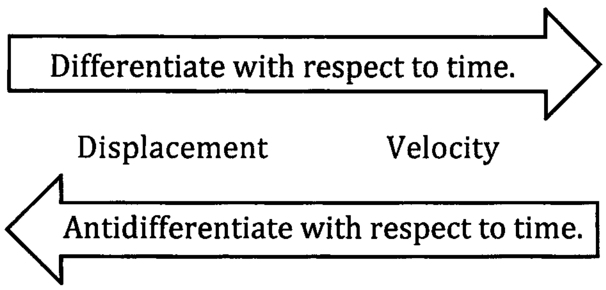
These rules apply for $0 \leq t \leq t_1$ where t_1 is when the particles collide.
(a) Find t_1.
(b) Find how far from O the collision occurs.
(c) Which of the following occur when $t = t_1$?
A catches up with B, B catches up with A, A and B collide "head-on".

Displacement from velocity.

From our understanding of antidifferentiation, or integration, as the reverse of differentiation it follows that integrating velocity with respect to time will give displacement.

i.e. $$x = \int v \, dt$$

Thus:



Remember that antidifferentiation, or integration, will introduce a constant which, given sufficient information, may be determined.

Example 4

A particle travels along a straight line with its velocity at time t seconds given by v m/sec where $v = 3t^2 + 2$.

The initial displacement of the particle from a point O on the line is ten metres.

Find the displacement from O when $t = 5$.

If $v = 3t^2 + 2$ then $\dfrac{dx}{dt} = 3t^2 + 2$

Thus $$x = \int (3t^2 + 2)\, dt$$
$$= t^3 + 2t + c$$

We know that initially, i.e. when $t = 0$, $x = 10$.

$\therefore$ $$10 = (0)^3 + 2(0) + c$$

i.e. $$c = 10.$$

Thus $$x = t^3 + 2t + 10$$

$\therefore$ When $t = 5$, $$x = 5^3 + 2(5) + 10$$
$$= 145.$$

When $t = 5$ the displacement from O is 145 metres.

Exercise 8B

Questions 1 to 6 all involve rectilinear motion with x metres and v m/s being the displacement and velocity of a body respectively, relative to an origin O, at time t seconds.

1. If $v = 6t^2 + 4$ find the displacement when $t = 2$ given that for $t = 1, x = 5$.

2. If $v = 10$ find the displacement when $t = 2$ given that for $t = 1, x = 24$.

3. If $v = 10t - 23$ find the times when the body is at the origin given that when $t = 1$, $x = -6$.

4. If $v = t(t + 2)$ and when $t = 3, x = 13$, find the initial displacement.

5. If $v = 6t - 18$ find the velocity when the body is at the origin, O, given that at the time $t = 3, x = -3$.

6. If $v = 4t - 10$, and when $t = 0$ the body is at the origin, O, find
   (a) when the body is next at O,
   (b) the displacement of the body when the velocity is zero.

7. A particle travels along a straight line with its velocity at time t seconds given by v m/s where $v = 4t + 1$. Find the distance travelled in the fifth second.

8. A particle travels along a straight line with its velocity at time t seconds given by v m/s where $v = 9 - 2t$. Find the distance travelled in the fifth second.

Miscellaneous Exercise Eight.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. For each of the following, without using a calculator, write the coordinates of the point where the graph meets the y-axis.
   (a) $y = 2x^4 + x^3 + 3x^2 - 2x - 6$
   (b) $y = 15 + 2x - 7x^2 - 2x^3$
   (c) $y = \dfrac{12}{x + 2}$
   (d) $y = \sqrt{3x + 9}$
   (e) $y = \dfrac{2x + 8}{x - 2}$
   (f) $y = (x - 1)(x + 1)(x - 3)(x + 4)$

2. For each of the following, without using a calculator, write the coordinates of the point(s) where the graph meets the x-axis.
   (a) $y = 2x - 6$
   (b) $y = 6 + 2x$
   (c) $y = x^2 - 9$
   (d) $y = (2 - x)(x - 7)$
   (e) $y = (x - 3)(x + 2)(2x - 7)$
   (f) $y = (x - 1)(x + 1)(x - 3)(x + 4)$

Express each of questions 3 to 14 as a power of 2. (I.e in the form 2^n.)

3. $2 \times 4 \times 8 \times 16 \times 32$
4. $\frac{1}{16}$
5. $\frac{1}{2 \times 4 \times 8}$
6. $0{\cdot}5$
7. $0{\cdot}25$
8. $2^7 \times 2^3$
9. $2^7 \times 2^3 \times 2 \times 4$
10. $2^7 \times 2^3 \div 16$
11. 8^2
12. $2 + 2 + 2 + 2$
13. 1
14. $(2^6 \div 2^2)^2$

Determine the value of *n* in each of questions 15 to 23.

15. $a^4 \div a^9 = a^n$
16. $a^9 \div a^n = a^4$
17. $a^n \div a^9 = a^4$
18. $(a^2)^3 = a^n$
19. $(\sqrt{a})^n = a^5$
20. $\sqrt{a} \times \sqrt[3]{a} = a^n$
21. $(a^2)^n \times a = a^7$
22. $\frac{a^9 \div a^n}{a^3 \times a^2} = a^3$
23. $\frac{\sqrt{a^5}}{\sqrt{a}} = a^n$

State the first five terms for each of the sequences given in numbers 24 to 29.

24. $T_{n+1} = T_n + 6,\ T_1 = 15.$
25. $T_{n+1} = T_n - 7,\ T_1 = 100.$
26. $T_n = 5T_{n-1},\ T_1 = 4.$
27. $T_{n+1} = 4T_n,\ T_3 = 96.$
28. $\ldots,\ T_6 = 243,\ T_7 = 249,\ T_8 = 255,\ T_9 = 261,\ T_{10} = 267,\ \ldots$
29. $\ldots,\ T_7 = 2916,\ T_8 = 4374,\ T_9 = 6561,\ T_{10} = 9841{\cdot}5,\ T_{11} = 14762{\cdot}25,\ \ldots$

30. If the angles of a triangle are in Arithmetic Progression show that one of the angles must be 60°.

31. Determine $\frac{dy}{dx}$ for each of the following.

    (a) $y = 3x^2$ (b) $y = 1 + 5x^3$

    (c) $y = 0{\cdot}5x^2 + 3x - 2$ (d) $y = (1 + 3x)(5x - 2)$

    (e) $y = (1 + 3x)^2$ (f) $y = (1 + x)(1 - x)$

32. Differentiate each of the following with respect to x.

    (a) $7x^2$ (b) $2x^3 + 5$

    (c) $3x^4 + x^3 - 5x^2 + 9x - 2$ (d) $(3x - 2)(x^2 + 1)$

33. Find the gradient of $y = 2x^3 + x$ at the point $(-1, -3)$.

34. If $f(x) = 3x^2 + 2x + 5$ determine (a) $f(1)$, (b) $f(-1)$, (c) $f'(x)$, (d) $f'(2)$.

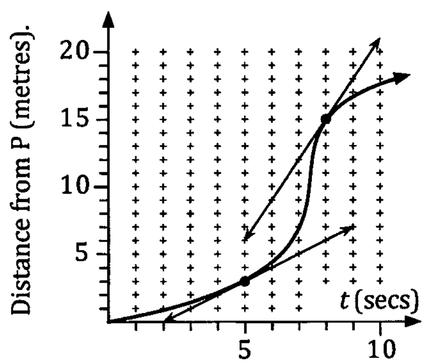
35. Find the point(s) on the following curves where the gradient is as stated
(a) $y = 5x^2$. Gradient of 5. (b) $y = 5x^3$. Gradient of 60.
(c) $y = x^2 + 3x$. Gradient of 7. (d) $y = x^3 - 3x^2$. Gradient of 24.

36. Determine the gradient of (a) $y = \frac{12}{x^2}$ at (2, 3), (b) $y = \frac{6}{\sqrt{x}}$ at (4, 3).

37. At what point on the quadratic $y = x^2 - 4x - 1$ is the tangent parallel to the straight line $y = -2x + 5$?

38. Find the equation of the line tangential to the curve $y = x^2$ and perpendicular to the line $2y = x + 6$.

39. An object leaves a point P and travels directly away from P, in a straight line, with its distance from P, in metres, t seconds later, as given by the curve in the graph on the right. The graph also shows the tangents to the curve at $t = 5$ and $t = 8$.
Use the graph to determine
(a) the average speed of the object for the interval $t = 0$ to $t = 5$,
(b) the average speed of the object for the interval $t = 5$ to $t = 8$,
(c) the speed of the object when $t = 5$,
(d) the speed of the object when $t = 8$.



40. The displacement of a body from an origin O, at time t seconds, is x metres where
$$x = 3t^2 - 12t + 1.$$
(a) Find the initial velocity of the body.
(b) Find the initial speed of the body.
(c) The value of t for which the body has a velocity of 3 m/s.
(d) The values of t for which the body has a speed of 3 m/s.

41. The graph of $y = ax^2 + bx - 2$ passes through the point (−3, 10) and its gradient at that point is −13. Find a and b and determine the coordinates of the point on the curve where the gradient is −1.

42. Find the equation of the tangent to $y = x - \frac{4}{x}$ at the point (4, 3).

43. Given that $f(x) = ax^3 + bx^2$, $f(2) = -4$ and $f'(3) = 99$, find $f(x)$, $f(3)$ and $f'(2)$.

44. For $f(x) = 2x^3 - 5x + 3$, and using your calculator if you wish, determine
(a) $f(12)$, $f(17)$ and $f(49)$. (b) $f'(12)$, $f'(17)$ and $f'(49)$.

45. Find the following indefinite integrals

(a) $\int 60\,dx$ (b) $\int 60x\,dx$

(c) $\int 60x^2\,dx$ (d) $\int 60x^3\,dx$

(e) $\int 60x^4\,dx$ (f) $\int 60x^5\,dx$

(g) $\int (8x^3 - 15x^2 + 2)\,dx$ (h) $\int (4 - 3x + 2x^2 - x^3)\,dx$

(i) $\int (x - 3)(x + 3)\,dx$ (j) $\int 24x^2(2x - 1)\,dx$

46. Find y as a function of x given that

(a) $\frac{dy}{dx} = 4x - 3$ and when $x = 2, y = 5$.

(b) $\frac{dy}{dx} = 6x^2 - 2x + 4$ and when $x = -1, y = 0$.

(c) $\frac{dy}{dx} = 8x^3 - 12x^2 - 4x + 11$ and when $x = 2, y = 4$.

47. Given that $\frac{dy}{dx} = 10x^4 - 6x + 1$, and that when $x = -1$, $y = 4$, find y when $x = 2$.

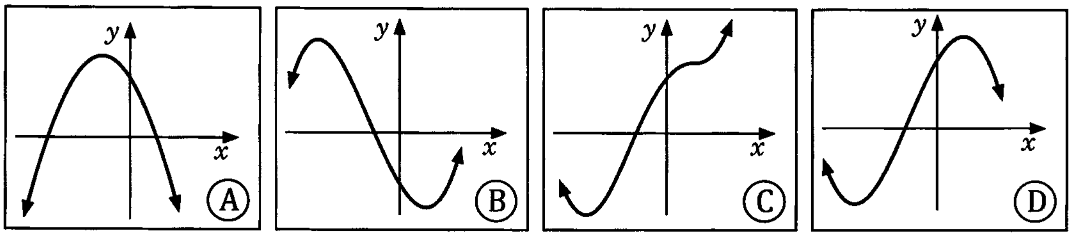
48. The radius of a sphere is increasing in such a way that the volume, V cm^3, at time t seconds is given by

$$V = 7500 + 5400t - 450t^2 + \frac{25t^3}{2} \quad \text{for } 0 \le t \le 12.$$

Calculate
(a) the volume when $t = 0$,
(b) the volume when $t = 12$,
(c) an expression for the instantaneous rate of change in the volume with respect to time,
(d) the rate at which the volume is increasing (in cm^3/sec) when $t = 2$, when $t = 4$ and when $t = 10$.

49. For $y = 3x^2 + 2x + 1$ determine:
(a) By how much y changes when x changes from $x = 0$ to $x = 10$.
(b) The average rate of change in y, per unit change in x, when x changes from $x = 0$ to $x = 10$.
(c) The instantaneous rate of change of y, with respect to x, when $x = 0$.
(d) The instantaneous rate of change of y, with respect to x, when $x = 10$.

50. One of the graphs A to D shown below has $\frac{dy}{dx} = (1 - x)(x + 3)$. Which one?



51. A company manufacturing toys wishes to launch a new product but is not sure what price to charge for it. If they charge more they will make more profit on each one they sell but will they sell less because of the higher price?
The minimum they anticipate charging for each of the new toys is \$15.
Market research indicates that if they charge \$(15 + x) the "demand curve", (likely sales figure, S thousand, plotted against x) has equation:

$$S = 6 + 6x - x^2.$$

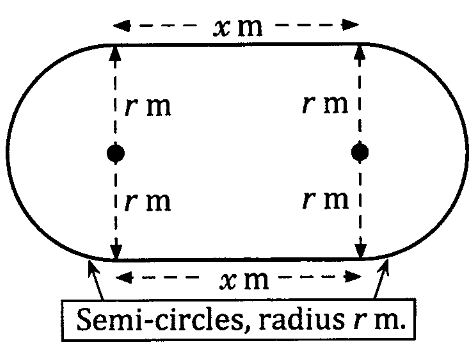
Clearly showing your use of calculus, determine the price the company should charge for each toy to maximise likely sales and find what this maximum sales figure would be.

52. An object moves in a straight line such that its distance, x metres, from some point S on the line, at time t seconds, is given by

$$x = 0{\cdot}25t^3 + 3$$

Find
(a) the distance the object is from S when $t = 2$,
(b) the distance the object is from S when $t = 6$,
(c) the average rate of change in the distance from S from $t = 2$ to $t = 6$,
(d) the instantaneous rate of change of x per second when $t = 2$,
(e) the instantaneous rate of change of x per second when $t = 6$.

53. A group of go-cart enthusiasts wish to purchase some land for a go cart track (see diagram). The perimeter of the track is to be 500 metres.
(a) Find an expression for x in terms of r (and π).
(b) Find an expression for A, the area enclosed by the track, in terms of r.
(c) Use differentiation to show that for A to be a maximum the track should be made circular.



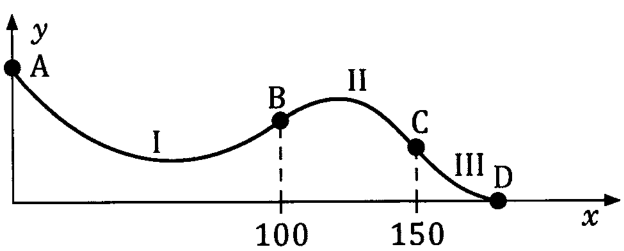
54. On 1st January 2015 Nadine opens an account by depositing \$5000 in an account earning interest at 6·8 % compounded annually.
On every 1st January thereafter Nadine adds a further \$800 to the account.
If the interest remains at 6·8 % throughout how much is the account worth when Nadine closes it on 31st December 2025?

55. The organisers of a scheme to raise funds to send a group of dancers to an interstate competition plan to sell boxes of cakes. They estimate that if they charge \$3 per box they will sell 120 boxes. For each five cents that they raise the price per box they expect their sales to decrease by one box.
Suppose they raise the price by "x" lots of five cents i.e. by $5x$ cents.
The price per box will then be $\$(3 + 0{\cdot}05x)$.
   (a) Write down an expression, in terms of x, for the number of boxes they would then expect to sell.
   (b) Write down a revenue function, $R(x)$, for the total revenue (i.e total income before costs are deducted) for selling this number of boxes at this price.
   (c) What price should they sell each box for to maximise the revenue?
   (d) What would this maximum revenue be?

   It costs the organisers \$2 to produce each box of cakes.
   (e) Write down a profit function, $P(x)$, for the total profit they could expect for selling the cakes at $\$(3 + 0{\cdot}05x)$ per box.
   (f) For what price should they sell each box to maximise their profit?
   (g) What would this maximum profit be?

56. The displacement of a body from an origin O, at time t seconds, is x metres where x is given by: $x = 10t - t^2 + 4$
Find (a) the displacement when $t = 7$,
   (b) the value of t when the body is a rest,
   (c) the distance the body travels from $t = 1$ to $t = 7$.

57. The final sections of a big dipper ride at an amusement park has the shape shown on the right.
The sections, A → B, B → C and C → D, are quadratic functions.
At B and at C, the points where one quadratic function flows into the next, there are no "gaps" between the functions and the gradient at the end of one function is the same as the gradient at the beginning of the next.



For section I, i.e. A → B the equation is $y = 0{\cdot}01x^2 - 1{\cdot}2x + 50$.
For section II, i.e. B → C the equation is $y = ax^2 + bx - 250$.
For section III, i.e. C → D the equation is $y = cx^2 + dx + 605$.
Point A has coordinates $(0, e)$. Point B has coordinates $(100, f)$.
Point C has coordinates $(150, g)$. Point D has coordinates $(h, 0)$.
Determine a, b, c, d, e, f, g and h.